PMM U.S.S.R.,Vol.44,pp.776-781
Copyright Pergamon Press Ltd.1981.Printed in U.K.
0021-8928/81/6 0776 $7.50/0

UDC 539.3:534.1

ON THE PARAMETRIC RESONANCE DENSITY*

V. V. BOLOTIN

The relation between the asymptotic density of the eigenfrequencies of elastic systems and their reactions to parametric vibrational effects is examined. Main attention is paid to the relationship between the density of simple parametric resonances at which one of the eigenmodes of the system is mostly excited, and the density of the combined parametric resonances accompanied by a pairwise interaction of the eigenmodes.

1. Many elastic systems possess a sufficiently compact eigenfrequency spectrum, especially in the high-frequency domain. The approximate nature of the spectrum, the asymptotic density of the eigenfrequencies, approximately equal to the number of eigenfrequencies per unit frequency band, is of interest for such systems. The higher the asymptotic density of the eigenfrequencies, the more intense the dynamic reaction of the system to external vibrational effects with slowly varying frequencies, as well as the stationary widebanded random effects /1/.

Let us consider an elastic system whose small vibrations are described by the operator equation

$$A(t,\mu)\mathbf{u}^{\cdot\cdot} + \beta B(t,\mu)\mathbf{u}^{\cdot} + C(t,\mu)\mathbf{u} = 0 \tag{1.1}$$

Here $\mathbf{u}(\mathbf{x},t)$ is the displacement vector in the system V dependent on the coordinate $\mathbf{x} \in V \subset R^m$ and the time $t \in [t_0, \infty)$. This vector is an element of a certain space; we shall assume that it is a real separable Hilbert space and that A, B and C are linear operators in this space which are continuous functions of the time t and a nonnegative parameter μ, where these operators become independent of the time as $\mu \to 0$. In the majority of applications it turns out that the operators A, B and C are symmetric and positive-definite for all the values of t and μ under consideration; moreover, the operator C^{-1} is completely continuous. The operator A is the gradient of the system kinetic energy, βB is the gradient of a dissipative function, and C is a gradient of the generalized potential energy. The non-negative parameter β characterizes the energy dissipation level in the system. For $\beta = \mu = 0$ equation (1.1) describes the free vibrations in an appropriate conservative system. The vibration eigenmodes $\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \ldots$ are defined as eigen elements of the equation

$$(C - \omega^2 A)\varphi = 0 \tag{1.2}$$

and the corresponding eigenfrequencies as positive square roots $\omega_1, \omega_2, \ldots$ of the eigenvalues ω^2 of this equation.

Let the operator coefficients in (1.1) be periodic functions of the time with period T and frequency $\omega = 2\pi / T$. The points in the parameter space describing the system properties (including the frequency of excitation ω) can be separated into two sets according to the stability or instability criterion for the trivial solution $\mathbf{u}(\mathbf{x},t) \equiv 0$. For small β and μ the instability domains have the form of wedges adjacent to the ω axis /2/. The frequency relationships characterizing the disposition of these wedges are called parametric resonances. One distinguishes between simple (1.3) and combined (1.4) parametric resonances

$$\omega = 2\,\omega_k / p \quad (k = 1, 2, \ldots; \; p = 1, 2, \ldots) \tag{1.3}$$

$$\omega = (\omega_j + \omega_k) / p \quad (j, k = 1, 2, \ldots; \; j \neq k; \; p = 1, 2, \ldots) \tag{1.4}$$

The latter are accompanied by pairwise interaction of the eigen modes $\varphi_j(\mathbf{x})$ and $\varphi_k(\mathbf{x})$.

In some well-studied particular cases /2/, equation (1.1) reduces to a countable set of ordinary differential equations with periodic coefficients in each generalized coordinate of the system after decomposition into the coordinate basis $\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \ldots$. In these cases, only simple resonances (1.3) are possible in the system, and this case will not be considered here. In the general case, all the generalized system coordinates interact so that almost all combination resonances (1.4) are realized.

Let us note that the requirement of symmetricity of the operator coefficients is essential in (1.1). If this requirement is not imposed, then instead of the combination resonances of the type (1.4), combination resonances of difference type may be encountered near the frequency relationships

*Prikl.Matem.Mekhan.,44,No.6,1087-1094,1980

$$\omega = |\omega_j - \omega_k| / p \ (j, k = 1, 2, \ldots; \ j \neq k; \ p = 1, 2, \ldots) \quad (1.5)$$

With the exception of singular cases, the density of the combination resonances (1.4) is higher than the density of the simple resonances (1.3). Let us assume that the coordinate realization of the Hilbert space is truncated to an n-space, i.e., the first n eigenmodes are taken into account in the computation. Then n simple resonances of the class p and $C_n^2 = \frac{1}{2} n (n-1)$ combination resonances of the same class should be expected. For instance, for $n = 100$ we will have $C_n^2 = 4950$. But this preliminary deduction still says nothing about the disposition of the resonances on the axes of the exciting frequencies ω.

Let the eigenfrequency spectrum of the system be sufficiently compact, so that the generalized characteristic of the spectrum, the asymptotic density of the eigenfrequencies /3/ would have meaning. Let $\nu(\omega)$ denote this characteristic which is asymptotically equal to the number of eigenfrequencies per unit frequency band. We here have in mind the asymptotic in that parameter (or group of parameters) whose tendency to zero makes the eigenfrequency spectrum as compact as desired. In thin and thin-walled elastic systems the ratio of the characteristic thickness to the characteristic length or the characteristic radius of curvature is such a parameter. By analogy with the asymptotic density of the eigenfrequencies, we introduce the asymptotic density of the parametric resonances, which are approximately equal to the number of resonance relationships of the type (1.3) or (1.4) per unit band of exciting frequencies ω. The asymptotic density $\nu_s^{(p)}(\omega)$ of simple resonances of the class p, i.e., those to which the natural number p in (1.3) corresponds, is evidently defined as

$$\nu_s^{(p)}(\omega) = \frac{p}{2} \nu\left(\frac{p\omega}{2}\right) \quad (1.6)$$

The density of all the simple resonances is determined by summing the densities over all classes p

$$\nu_s(\omega) = \sum_{p=1}^{\infty} \nu_s^{(p)}(\omega) \quad (1.7)$$

This density is not of noticeable interest for applications since it has meaning only for systems without dissipation. For dissipation different from zero the minimal value of the parameter μ for which instability becomes possible is on the order of $\beta^{1/p}$ /2/. Therefore, resonances of the class $p = 1$ are the most dangerous.

Let $\nu_c^{(p)}(\omega)$ denote the asymptotic density of combination resonances. This density can already not be expressed in terms of the eigenfrequency density by means of a simple relationship of the type (1.6). To calculate it we used a method analogous to that already used repeatedly in problems to estimate the eigenfrequency density /3–6/.

2. For a sufficiently broad class of elastic systems there are asymptotic estimates of the eigenfrequencies which depend weakly on the type of boundary conditions in the high-frequency domain /7,8/. Under these conditions the eigenfrequencies are ordered by using a certain wave vector $\mathbf{k}$ that takes on continuous values from the positive sector $K \subset R^m$, and one cell $\Delta\mathbf{k} = \Delta k_1 \Delta k_2 \ldots \Delta k_m$ corresponds to each eigenfrequency in the domain K. Here m is the dimensionality of the elastic system under consideration. Let $\omega = \Omega(\mathbf{k})$ denote the asymptotic dependence of the eigenfrequencies on the wave vector $\mathbf{k}$. Then for the number of eigenfrequencies not exceeding a given value ω we have the asymptotic estimate /3/

$$N(\omega) \sim \int\limits_{\Omega(\mathbf{k}) < \omega} \frac{d\mathbf{k}}{\Delta\mathbf{k}(\mathbf{k})}, \quad d\mathbf{k} = dk_1 dk_2 \ldots dk_m \quad (2.1)$$

If the function $N(\omega)$ is differentiable in a certain interval, then its derivative $\nu(\omega)$ has the meaning of an asymptotic density of the eigenfrequencies. Precisely this density was used in (1.6).

Let us estimate the number of combined resonances (1.4) for which the exciting frequency does not exceed the given value of ω. This number is estimated asymptotically as

$$N_c^{(p)}(\omega) \sim \frac{1}{2} \iint\limits_{\Lambda(p\omega)} \frac{d\mathbf{k} d\mathbf{k}'}{\Delta\mathbf{k}(\mathbf{k}) \Delta\mathbf{k}(\mathbf{k}')} \quad (2.2)$$

where the integration is over the subset $\Lambda(p\omega)$ of the direct product $K \times K'$ of two spaces of wave numbers

$$\Lambda(p\omega) = \{\mathbf{k} \in K, \mathbf{k}' \in K' : \Omega(\mathbf{k}) + \Omega(\mathbf{k}') < p\omega\} \quad (2.3)$$

The coefficient 1/2 in (2.2) is introduced in order not to take account of the same resonance twice, to which commutation of the subscripts in the frequencies ω_j and ω_k would correspond in (1.4). In a number of problems it can be assumed that $\Delta\mathbf{k} = \text{const}$ over the whole range of wave vector variation. Everywhere that the functions $N_c^{(p)}(\omega)$ is differentiable, the asymptotic density $\nu_c^{(p)}(\omega)$ of the parametric resonances, equal to the derivative of $N_c^{(p)}(\omega)$, has meaning. By analogy with (1.7), the density of all the combined resonances, the combined density of all the parametric resonances, etc. are introduced. If all the combined resonances belong to the difference type, then it is sufficient to replace the range of integration in (2.2) by

$$\Lambda(p\omega) = \{\mathbf{k} \in K, \mathbf{k}' \in K' : |\Omega(\mathbf{k}) - \Omega(\mathbf{k}')| < p\omega\}$$

Unfortunately, in applications where equations of the type (1.1) are encountered with nonsymmetric operators, only part of the combined resonances ordinarily has the form (1.5). Hence, without a special analysis of the specific system it is difficult to indicate what part of the eigenfrequency spectrum will be related to resonances of the sum type and what part to resonances of the difference type.

3. Let us exhibit the application of the general relationships (2.1) — (2.3) by two simple examples. Let us consider a rectilinear elastic rod of constant cross-section F and length l whose bending vibrations are periodically excited in time by an axial force with frequency ω. If the ends of the rod are hinge-supported, then all the generalized coordinates are separated during the passage over to a coordinate realization of Hilbert space so that no combined resonances occur. But, for instance, if one end of the rod is clamped, while the other is free, then all the generalized coordinates turn out to be related. For arbitrary boundary conditions, we have the following estimate for the eigenfrequencies

$$\omega \sim k^2 \left(\frac{EJ}{\rho F}\right)^{1/2} \tag{3.1}$$

Here EJ is the rod bending stiffness, and ρ is the material density. Noting that the size of one cell on the half-axis $k > 0$ of the wave numbers is $\Delta k = \pi / l$, we obtain the known formula for the asymptotic density of the eigenfrequencies /2/

$$\nu(\omega) = \frac{1}{2(\omega\omega_0)^{1/2}}, \quad \omega_0 = \frac{\pi^2}{l^2}\left(\frac{EJ}{\rho F}\right)^{1/2} \tag{3.2}$$

For the example under consideration the general formula (2.2) becomes

$$N_c^{(p)}(\omega) \sim \frac{1}{2(\Delta k)^2} \iint\limits_{k^2+k'^2<r^2(p\omega)} dk\, dk', \quad r^2(p\omega) = \frac{p\omega}{\omega_0^2}(\Delta k)^2$$

Elementary computations yield formulas for the asymptotic density of the parametric resonances

$$\nu_s^{(p)}(\omega) = \left(\frac{p}{8}\right)^{1/2} \frac{1}{(\omega_0\omega)^{1/2}}, \quad \nu_c^{(p)}(\omega) = \frac{\pi p}{8\omega_0} \tag{3.3}$$

Comparing these formulas we see that the density of resonances of a different kind behaves differently as ω grows. If the density of simple resonances is rarefied as ω grows (because of rarefaction in the eigenfrequency spectrum), then the asymptotic density of the combined resonances remains constant in the whole frequency range.

For the second example, we take an elastic plate of constant thickness h, material density ρ, and cylindrical stiffness D. Let the plate be loaded by periodic forces with frequency ω in the middle plane. We consider bending parametrically-excited plate vibrations. We will consider the plate to be rectangular with sides a_1 and a_2 although the final results are apparently not related to this constraint and refer to a plate of arbitrary planform but the same area. The asymptotic expression for the eigenfrequencies has the form /7/

$$\omega \sim (k_1^2 + k_2^2)\left(\frac{D}{\rho h}\right)^{1/2} \tag{3.4}$$

and the size of one cell is $\Delta k = \pi^2/(a_1 a_2)$. The asymptotic density of the eigenfrequencies is determined by the known expression /2/ first obtained by Courant

$$\nu(\omega) = \frac{1}{\omega_0}, \quad \omega_0 = \frac{4\pi}{a_1 a_2}\left(\frac{D}{\rho h}\right)^{1/2} \tag{3.5}$$

It follows from (3.5) that the asymptotic density of the simple parametric resonances is constant in the whole band of exciting frequencies. If the plate is supported along the whole contour, and the load in the middle plane is distributed uniformly along the sides of the plate, then only simple resonances are excited. If the plate is clamped around the contour then the eigenmodes are separated into several groups according to symmetry classes, within each of which combination resonances are possible. In the most general case, almost all combined resonances are excited; we examine precisely this case henceforth.

Application of the general formula (2.2) yields

$$N_c^{(p)}(\omega) \sim \frac{1}{2(\Delta k)^2} \iint\limits_{|\mathbf{k}|^2+|\mathbf{k}'|^2<r^2(p\omega)} d\mathbf{k}\, d\mathbf{k}', \quad r^2(p\omega) = \frac{4p\omega}{\pi\omega_0}\Delta k \tag{3.6}$$

where the integral is the volume of 1/16 -th of a four-dimensional sphere of radius $r(p\omega)$. Hence $N_c^{(p)}(\omega) \sim p^2\omega^2/(4\omega_0^2)$, and we arrive at the following formulas for the asymptotic densities of the parametric resonances

$$\nu_s^{(p)}(\omega) = p/(2\omega_0),\ \nu_c^{(p)}(\omega) = p^2\omega/(2\omega_0^2) \tag{3.7}$$

Comparing these formulas, we arrive at the conclusion already obtained earlier in the problem of parametric resonances of a rod: the density of the combined resonances is higher than the density of the simple resonances (with the exception of the initial segment of the spectrum where these estimates are not applicable). The density of combined resonances for plates hence grows asymptotically linearly as the exciting frequency grows.

4. More interesting problems are associated with parametrically excited vibrations of thin elastic shells for which the existence of singularities in the eigenfrequency asymptotic density has been detected /3/. These singularities correspond to shrinkage of the spectrum in the neighborhood of certain eigenfrequencies with clear mechanical meaning. An elevated response of the shells to vibrations, especially wideband random effects /1/ should be expected around the asymptotic condensation points.

Let us consider a thin elastic spherical panel of thickness h, middle surface radius R and cylindrical stiffness D. For definiteness, we assume the panel to be rectangular with sides a_1 and a_2. Forces periodic in time with frequency ω act in the panel middle surface. We shall consider the parametric excitation of primarily bending vibrations since the eigenfrequencies to which the membrane strains correspond primarily are in the remote part of the spectrum for thin shells. Let us assume that the boundary conditions and (or) the force distribution in the middle surface are such that almost all the parametric resonances (1.3) and (1.4) are excited. For the eigenfrequencies we have the asymptotic expression /7/

$$\omega = [\omega_R^2 + (k_1^2 + k_2^2)^2 D/(\rho h)]^{1/2}, \quad \omega_R = (E/\rho)^{1/2}/R \tag{4.1}$$

Remarking that $\Delta\mathbf{k} = \pi^2/(a_1a_2)$, and using the notation from (3.5) for the characteristic frequency of the bending vibrations ω_0, we arrive at a formula for the asymptotic density of the eigenfrequencies

$$(\omega) = \begin{cases} 0 & (\omega < \omega_R) \\ \dfrac{\omega}{\omega_0(\omega^2-\omega_R^2)^{1/2}} & (\omega > \omega_R) \end{cases} \tag{4.2}$$

Formula (4.2), which was first obtained in /3/, discloses an asymptotic condensation point at the frequency $\omega = \omega_R$. This frequency corresponds to radial membrane vibrations. Formula (4.2) remains applicable for panels of other shape as well as for a closed spherical shell. In the latter case it is sufficient to replace the area of the rectangular panel a_1a_2 by $4\pi R^2$ in the expression for ω_0. The density of the simple resonances is evaluated later by means of (1.6) with (4.2) taken into account.

$$\nu_s^{(p)}(\omega) = \begin{cases} 0 & (\omega < 2\omega_R/p) \\ \dfrac{p}{2\omega_0}\left(1 - \dfrac{4\omega_R^2}{p^2\omega^2}\right)^{-1/2} & (\omega > 2\omega_R/p) \end{cases} \tag{4.3}$$

Going over to the combined resonances, we note that, taking account of (4.1), the domain of integration (2.3) will be

$$\Lambda(p\omega) = \{\mathbf{k} \in K, \mathbf{k}' \in K' : (\omega_R^2 + \omega_0^2 |\mathbf{k}|^4/k_0^4)^{1/2} + (\omega_R^2 + \omega_0^2 |\mathbf{k}'|^4/k_0^4)^{1/2} < p\omega\}$$

Here $k_0^2 = 4\pi/(a_1a_2)$. To evaluate the integral in (2.2), we go over to new coordinates r_1, $r_2 > 0$ and $\theta_1, \theta_2 \in [0, \pi/2]$.

$$k_1 = k_0\sqrt{2r_1}\cos\theta_1, \quad k_1' = k_0\sqrt{2r_2}\cos\theta_2, \quad k_2 = k_0\sqrt{2r_1}\sin\theta_1, \quad k_2' = k_0\sqrt{2r_2}\sin\theta_2$$

We obtain a double integral ($\alpha = \omega_0/\omega_R$) in place of the quadruple integral

$$N_c^{(p)}(\omega) \sim 2 \iint\limits_{M(p\omega/\omega_R)} dr_1 dr_2, \quad M(z) = \{r_1 > 0, r_2 > 0; (1 + 4\alpha^2 r_1^2)^{1/2} + (1 + 4\alpha^2 r_2^2)^{1/2} < z\}$$

Still another integration is performed elementarily. Let us write its result down together with an analogous formula for the combined resonance density

$$N_c^{(p)}(\omega) \sim \frac{1}{2\alpha^2} J(z), \quad \nu_c^{(p)}(\omega) = \frac{p}{2\alpha^2\omega_R}\frac{dJ(z)}{dz} \tag{4.4}$$

where we have used the notation

$$J(z) = \int_1^{z-1} \frac{\sqrt{(z-y)^2-1}\, y\,dy}{\sqrt{y^2-1}}, \quad \frac{dJ(z)}{dz} = \int_1^{z-1} \frac{(z-y)\,y\,dy}{\sqrt{(y^2-1)[(z-y)^2-1]}} \tag{4.5}$$

$$z = p\omega/\omega_R$$

The integrals in the right sides of these formulas are reduced to elliptic integrals, however, the actual calculations are rather tedious. The integration limits in (4.5) turned out to be matched with the integrands so that the final results is expressed in terms of the complete elliptic integrals $\mathbf{K}(\cdot)$ and $\mathbf{E}(\cdot)$ of the first and second kinds. Introducing the notation $\zeta = (1-4/z^2)^{1/2}$, where $z > 2$, we write the result in the form

$$J(z) = \frac{z^2}{2}\left[\mathbf{E}(\zeta) - \frac{4}{z^2}\mathbf{K}(\zeta)\right], \quad \frac{dJ(z)}{dz} = z\left[\mathbf{E}(\zeta) - \frac{2}{z^2}\mathbf{K}(\zeta)\right] \tag{4.6}$$

Taking account of (4.4) and (4.6), we finally obtain

$$\nu_c^{(p)}(\omega) = \frac{p^2\omega}{2\omega_0^2}\left[\mathbf{E}\left(\sqrt{1-\frac{4\omega_R^2}{p^2\omega^2}}\right) - \frac{2\omega_R^2}{p^2\omega^2}\mathbf{K}\left(\sqrt{1-\frac{4\omega_R^2}{p^2\omega^2}}\right)\right] \tag{4.7}$$

Let us examine two limit cases. The first case is when $\omega \downarrow 2\omega_R/p$, which corresponds to resonances at eigenfrequencies in the neighborhood of the condensation point. This limit is finite

$$\lim_{\omega \downarrow 2\omega_R/p} \nu_c^{(p)}(\omega) = \frac{\pi p \omega_R}{4\omega_0^2}$$

i.e., there are no combined resonance condensation points in the neighborhood of the frequency $\omega = 2\omega_R/p$. At first glance, such a deduction can seem to be unexpected since there are condensation points near $\omega = 2\omega_R/p$ for simple resonances (they simply reproduce the eigenfrequency condensation point at $\omega = \omega_R$ in multiple respects). However, it is essential that the combined resonance density already be considerably higher for a quite moderate distance from the frequency $\omega = 2\omega_R/p$ than for simple resonances. The other limit case $\omega/\omega_R \to \infty$ corresponds to the passage from a hollow spherical panel to a flat plate. Here (4.7) goes over into (3.7).

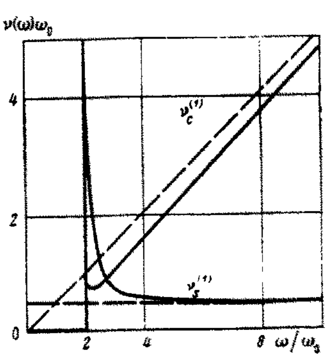

Fig.1

The results of calculations using (1.6), (3.5), (3.7), (4.2) and (4.7) are presented in the Fig.1 for $p = 1$. The continuous lines are constructed for shells with the relationship $\omega_R/\omega_0 = 1$ between the characteristic frequencies, while the dashed lines are for an analogous plate, i.e., for $\omega_R/\omega_0 \to 0$. As is seen from the graph, the asymptotic density of simple resonances for a spherical shell is greater than for the corresponding plate while the asymptotic density of the combined resonances is less. This is explained by the fact that the eigenfrequency spectrum for shells starts with the frequency ω_R so that a pair for the formation of combination resonances is initially not available for the eigenmodes. This is also the reason for the absence of singularities in the functions $\nu_c^{(1)}(\omega)$ for $\omega = 2\omega_R$. Let us also note that the ratio

$$\frac{\omega_R}{\omega_0} = \frac{a_1 a_2}{Rh}\frac{\sqrt{12(1-\mu^2)}}{\pi}$$

(μ is the Poisson's ratio) can vary within quite broad limits, and is on the order of R/h for $a_1 \sim a_2 \sim R$.

REFERENCES

1. BOLOTIN V.V., Random Vibrations of Elastic Systems. "Nauka", Moscow, 1979.

2. CHELOMEI V.N., ed. Vibrations in Engineering. Handbook in Six Volumes. Vol.1, Vibrations of Linear Systems. "Mashinostroenie", Moscow, 1978.

3. BOLOTIN V.V., On the density of the distribution of natural frequencies of thin elastic shells, PMM, Vol.27, No.2, 1963.

4. MOSKALENKO V.N., On the frequency spectra of natural vibrations of shells of revolution, PMM, Vol.36, No.2, 1972.

5. TOVSTIK P.K., On the vibrations frequency density of thin shells of revolution. PMM, Vol. 36, No.2, 1972.

6. KHROMATOV V.E., Properties of the spectra of thin circular cylindrical shells vibrating in the neighborhood of the membrane state of stress, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.2. 1972.

7. BOLOTIN V.V., Edge effect in the oscillations of elastic shells, PMM, Vol.24, No.5, 1960.

8. ZHINZHER N.I., Dynamic edge effects in orthotropic elastic shells. PMM, Vol.39, No.4,1975.

Translated by M.D.F.